

Hardness result for the total rainbow k -connection of graphs*

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Abstract

A path in a total-colored graph is called *total rainbow* if its edges and internal vertices have distinct colors. For an ℓ -connected graph G and an integer k with $1 \leq k \leq \ell$, the *total rainbow k -connection number* of G , denoted by $trc_k(G)$, is the minimum number of colors used in a total coloring of G to make G *total rainbow k -connected*, that is, any two vertices of G are connected by k internally vertex-disjoint total rainbow paths. In this paper, we study the computational complexity of total rainbow k -connection number of graphs. We show that it is NP-complete to decide whether $trc_k(G) = 3$.

Keywords: total rainbow k -connection number, computational complexity.

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1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty[2] for those not defined here. A set of internally vertex-disjoint paths are called *disjoint*. Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \rightarrow \{0, 1, \dots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored the same. A path in G is called a *rainbow path* if no two edges of the path are colored the same. The graph G is called *rainbow connected* if for any two vertices of G , there is a rainbow path connecting them. The *rainbow connection*

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number of G , denoted by $rc(G)$, is defined as the minimum number of colors that are needed to make G rainbow connected. If G is an ℓ -connected graph with $\ell \geq 1$, then for any integer $1 \leq k \leq \ell$, G is called *rainbow k -connected* if any two vertices of G are connected by k disjoint rainbow paths. The *rainbow k -connection number* of G , denoted by $rc_k(G)$, is the minimum number of colors that are required to make G rainbow k -connected. The concepts of rainbow connection and rainbow k -connection of graphs were introduced by Chartrand et al. in [6, 5], and have been well-studied since then. For further details, we refer the readers to the book [15].

Let G be a nontrivial connected graph with a *vertex-coloring* $c : V(G) \rightarrow \{0, 1, \dots, t\}$, $t \in \mathbb{N}$, where adjacent vertices may be colored the same. A path in G is called a *vertex-rainbow path* if no interval vertices of the path are colored the same. The graph G is *rainbow vertex-connected* if for any two vertices of G , there is a vertex-rainbow path connecting them. The *rainbow vertex-connection number* of G , denoted by $rvc(G)$, is the minimum number of colors used in a vertex-coloring of G to make G rainbow vertex-connected. If G is an ℓ -connected graph with $\ell \geq 1$, then for any integer $1 \leq k \leq \ell$, the graph G is *rainbow vertex k -connected* if any two vertices of G are connected by k disjoint vertex-rainbow paths. The *rainbow vertex k -connection number* of G , denoted by $rvc_k(G)$, is the minimum number of colors that are required to make G rainbow vertex k -connected. These concepts of rainbow vertex connection and rainbow vertex k -connection of graphs were proposed by Krivelevich and Yuster [11] and Liu et al. [16], respectively.

Liu et al. [17] introduced the analogous concepts of total rainbow k -connection of graphs. Let G be a nontrivial ℓ -connected graph with a *total-coloring* $c : E(G) \cup V(G) \rightarrow \{0, 1, \dots, t\}$, $t \in \mathbb{N}$, where $\ell \geq 1$. A path in G is called a *total-rainbow path* if its edges and interval vertices have distinct colors. For any integer $1 \leq k \leq \ell$, the graph G is called *total rainbow k -connected* if any two vertices of G are connected by k disjoint total-rainbow paths. The *total rainbow k -connection number* of G , denoted by $trc_k(G)$, is the minimum number of colors that are needed to make G total rainbow k -connected.

When $k = 1$, we simply write $trc(G)$, just like $rc(G)$ and $rvc(G)$. From Liu et al.[17], we have that $trc(G) = 1$ if and only if G is a complete graph, and $trc(G) \geq 3$ if G is not complete. If G is an ℓ -connected graph with $\ell \geq 1$, then $trc_k(G) \geq 3$ if $2 \leq k \leq \ell$, and $trc_k(G) \geq 2diam(G) - 1$ for $1 \leq k \leq \ell$, where $diam(G)$ denotes the diameter of G . In relation to $rc_k(G)$ and $rvc_k(G)$, they have $trc_k(G) \geq \max(rc_k(G), rvc_k(G))$. Also, if $rc_k(G) = 2$, then $trc_k(G) = 3$. If $rvc_k(G) \geq 2$, then $trc_k(G) \geq 5$.

The computational complexity of the rainbow connectivity and vertex-connectivity has been attracted much attention. In [4], Chakraborty et al. proved that deciding whether $rc(G) = 2$ is NP-Complete. Analogously, Chen et al.[8] showed that it is NP-complete to decide whether $rvc(G) = 2$. Motivated by [4, 8], we consider the computational complexity of computing the total rainbow k -connectivity $trc_k(G)$ of a graph G . For $k = 1$, Chen et al. recently gave reductions to prove that it is NP-complete to decide whether $trc(G) = 3$ in [7]. In this paper, we prove that for any fixed $k \geq 1$ it is NP-complete to decide whether $trc_k(G) = 3$. The reduction of our proof is different from that in [7].

2 Main results

In the following, we will show that deciding whether $trc_k(G) = 3$ is NP-complete for fixed $k \geq 1$.

Theorem 2.1. *Given a graph G , deciding whether $trc_k(G) = 3$ is NP-Complete for fixed $k \geq 1$.*

We first define the following three problems.

Problem 1. The total rainbow connection number 3.

Given: Graph $G = (V, E)$.

Decide: Whether there is a total coloring of G with 3 colors such that all the pairs $\{u, v\} \in (V \times V)$ are total rainbow k -connected?

Problem 2. The subset total rainbow k -connection number 3.

Given: Graph $G = (V, E)$ and a set of pairs $P \subseteq (V \times V)$, where P contains nonadjacent vertex pairs.

Decide: Whether there is a total-coloring of G with 3 colors such that all the pairs $\{u, v\} \in P$ are total rainbow k -connected?

Problem 3. The subset partial edge-coloring.

Given: Graph $G = (V, E)$ with a set of pairs $Q \subseteq V \times V$ where Q contains nonadjacent vertex pairs, and a partial 2-edge-coloring $\hat{\chi}$ for $\hat{E} \subset E$.

Decide: Whether $\hat{\chi}$ can be extended to a 3-total-coloring χ of G that makes all the pairs in Q total rainbow k -connected and $\chi(e) \notin \{\chi(u), \chi(v)\}$ for all $e = uv \in \hat{E}$?

In the following, we first reduce Problem 2 to Problem 1, and then reduce Problem 3 to Problem 2. Finally, Theorem 2.1 is completed by reducing 3-SAT to Problem 3.

Before proving Theorem 2.1, we need an useful result shown in [6].

Lemma 2.2. [6] *For every $k \geq 2$, $rc_k(K_{(k+1)^2}) = 2$. Furtherly, the following 2-edge coloring can make G rainbow k -connected. Let G_1, G_2, \dots, G_{k+1} be mutually vertex-disjoint graphs, where $V(G_i) = V_i$, such that $G_i = K_{k+1}$ for $1 \leq i \leq k+1$. Let $V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,k+1}\}$ for $1 \leq i \leq k+1$. Let G be the join of the graphs G_1, G_2, \dots, G_{k+1} . Thus $G = K_{(k+1)^2}$ and $V(G) = \cup_{i=1}^{k+1} V_i$. We assign the edge uv of G the color 0 if either $uv \in E(G_i)$ for some i ($1 \leq i \leq k+1$) or if $uv = v_{i,l}v_{j,l}$ for some i, j, l with $1 \leq i, j, l \leq k+1$ and $i \neq j$. All other edges of G are assigned the color 1.*

For $k = 1$, since $rc_1(K_{(k+1)^2})=1$, the above coloring surly makes G rainbow 1-connected.

Note that from the above coloring, for every vertex $v \in V(G)$, we have $d(v) = k^2 + 2k$, $2k$ edges incident with v colored with 0, and k^2 edges incident with v colored with 1.

Lemma 2.3. *Problem 2 \preceq Problem 1.*

Proof. Given a graph $G = (V, E)$ and a set of pairs $P \subseteq V \times V$ where P contains nonadjacent vertex pairs, we construct a graph $G' = (V', E')$ as follows. For every vertex $v \in V$, we introduce a new vertex set $V_v = \{x_{(v,1)}, x_{(v,2)}, \dots, x_{(v,(k+1)^2)}\}$, and for every pair $\{u, v\} \in (V \times V) \setminus P$, we introduce a new vertex set $V_{(u,v)} = \{x_{(u,v,1)}, x_{(u,v,2)}, \dots, x_{(u,v,(k+1)^2)}\}$. We set

$$V' = V \cup \{V_v : v \in V\} \cup \{V_{(u,v)} : \{u, v\} \in (V \times V) \setminus P\}$$

and

$$\begin{aligned} E' = & E \cup \{vx_{(v,i)} : v \in V, x_{(v,i)} \in V_v\} \\ & \cup \left\{ \{ux_{(u,v,i)}, vx_{(u,v,i)}\} : \{u, v\} \in (V \times V) \setminus P, x_{(u,v,i)} \in V_{(u,v)} \right\} \\ & \cup \{xx' : x, x' \in V' \setminus V\}. \end{aligned}$$

It remains to verify that G' is 3-total rainbow k -connected if and only if there is a total-coloring of G with 3 colors such that all the pairs $\{u, v\} \in P$ are total rainbow k -connected. In one direction, suppose that G' is 3-total rainbow k -connected. Notice that when G is considered as a subgraph of G' , no pair of vertices of G that appear in P has a path of length two in G' that is not fully contained in G . Then with this coloring, all the pairs $\{u, v\} \in P$ are total rainbow k -connected in G .

In the other direction, suppose that $\chi : V \cup E \rightarrow \{0, 1, 2\}$ is a total-coloring of G that makes all the pairs in P total rainbow k -connected. We now extend it to a total rainbow k -connection coloring $\chi' : V' \cup E' \rightarrow \{0, 1, 2\}$, $\chi'(x) = 2$ for all $x \in V' \setminus V$; $\chi'(v, x_{(v,i)}) = 1$ for all $v \in V$ and $x_{(v,i)} \in V_v$; $\chi'(u, x_{(u,v,i)}) = 0, \chi'(v, x_{(u,v,i)}) = 1$ for all $\{u, v\} \in (V \times V) \setminus P$ and all $x_{(u,v,i)} \in V_{(u,v)}$. The edges in $G'[V_v]$ or $G'[V_{(u,v)}]$ are colored with $\{0, 1\}$ as Lemma 2.2 for all $v \in V$ and all $\{u, v\} \in (V \times V) \setminus P$. Finally, the remaining uncolored edges are colored with 0. Now we show that G' is total rainbow k -connected under this coloring. For $\{u, v\} \in P$, the k disjoint total rainbow paths in G connecting u and v are also k -disjoint total rainbow paths in G' . For $\{u, v\} \in (V \times V) \setminus P$, $\{ux_{(u,v,1)}v, ux_{(u,v,2)}v, \dots, ux_{(u,v,k)}v\}$ are k disjoint total

rainbow paths. For $u \in V, v \in V' \setminus V$, if $v \notin V_u$, then $\{ux_{(u,1)}v, ux_{(u,2)}v, \dots, ux_{(u,k)}v\}$ are k disjoint total rainbow paths; if $v \in V_u$, from Lemma 2.2, we have $2k > k$ edges incident with v are colored with 0 in $G'[V_u]$. Suppose that $\{v_1, v_2, \dots, v_k\}$ are k vertices adjacent to v by these edges colored with 0, then $\{uv_1v, uv_2v, \dots, uv_kv\}$ are k disjoint total rainbow paths. For $\{x, x'\} \in (V_u \times V_u)$ or $(V_{(u,v)} \times V_{(u,v)})$, by Lemma 2.2, there are k disjoint total rainbow paths in $G'[V_u]$ or $G'[V_{(u,v)}]$ connecting u and v . For the remaining pairs $\{x, x'\}$, suppose w.l.o.g that $x \in V_u$ and $x' \in V_v (u \neq v)$. By Lemma 2.2, we have $k^2 > k$ edges incident with v are colored with 1 in $G'[V_v]$. Suppose that $\{v'_1, v'_2, \dots, v'_k\}$ are k vertices adjacent with v by these edges colored with 1, then $\{uv'_1v, uv'_2v, \dots, uv'_kv\}$ are k disjoint total rainbow paths. Hence χ' is indeed a total k -rainbow coloring of G' . \square

Lemma 2.4. *Problem 3 \preceq Problem 2.*

Proof. Since the identity of the colors does not matter, it is more convenient that instead of a partial 2-edge coloring $\hat{\chi}$ we consider the corresponding partition $\pi_{\hat{\chi}} = (\hat{E}_1, \hat{E}_2)$. For the sake of convenience, let $e = e^1e^2$ for $e \in (\hat{E}_1 \cup \hat{E}_2)$. Note that the ends of e may be labeled by different signs for $e \in (\hat{E}_1 \cup \hat{E}_2)$. Given such a partial 2-edge coloring $\hat{\chi}$ and a set of pairs $Q \subseteq (V \times V)$ where Q contains nonadjacent vertex pairs. Now we construct a graph $G' = (V', E')$ and define a set of pairs $P \subseteq (V' \times V')$ as follows. We first add the vertices

$$\{c, b_1, b_2\} \cup \left\{ \{c_e^j, d_e^j, f_e^j\} : j \in \{1, 2\}, e \in (\hat{E}_1 \cup \hat{E}_2) \right\}$$

and add the edges

$$\{b_1c, b_2c\} \cup \left\{ cc_e^j : j \in \{1, 2\}, e \in (\hat{E}_1 \cup \hat{E}_2) \right\} \cup \left\{ \{c_e^j f_e^j, c_e^j d_e^j, d_e^j e^j\} : e \in (\hat{E}_1 \cup \hat{E}_2) \right\}.$$

Now we define the set of pairs P .

$$P = Q \cup \{b_1, b_2\} \cup \left\{ \{b_i, c_e^j\} : e \in \hat{E}_i, i, j \in \{1, 2\} \right\} \\ \cup \left\{ \{f_e^j, c\}, \{f_e^j, e^j\}, \{d_e^j, c_e^j\}, \{d_e^j, e^{(3-j)}\} : j \in \{1, 2\}, e \in (\hat{E}_1 \cup \hat{E}_2) \right\}.$$

Then we secondly add the new vertices

$$\left\{ \{g_{(u,v,2)}, g_{(u,v,3)}, \dots, g_{(u,v,k)}\} : \{u, v\} \in P \setminus Q \right\}$$

and add the new edges

$$\left\{ \{ug_{(u,v,2)}v, ug_{(u,v,3)}v, \dots, ug_{(u,v,k)}v\} : \{u, v\} \in P \setminus Q \right\}.$$

On one hand, if there is a 3-total-coloring of χ of G that makes all the pairs in Q total rainbow k -connected which extends $\pi_{\hat{\chi}} = (\hat{E}_1, \hat{E}_2)$ and $\chi(e) \notin \{\chi(e^1), \chi(e^2)\}$ for all $e = e^1e^2 \in \hat{E}$, then we give a total-coloring χ' of G' as follows. Suppose w.l.o.g that \hat{E}_1 are colored with 0, and \hat{E}_2 are colored with 1. $\chi'(v) = \chi(v)$, and $\chi'(e) = \chi(e)$ for all $v \in V, e \in E$; $\chi'(v) = 2$ for all $v \in V' \setminus V$; $\chi'(b_1c) = 1$, and $\chi'(b_2c) = 0$; $\chi'(c_e^je^j) = \chi'(c_e^jc) = 0$, and $\chi'(d_e^je^j) = \{1, 2\} \setminus \chi(e^j)$ for all $e \in \hat{E}_1$; $\chi'(c_e^je^j) = \chi'(c_e^jc) = 1$, and $\chi'(d_e^je^j) = \{0, 2\} \setminus \chi(e^j)$ for all $e \in \hat{E}_2$; $\chi'(ug_{(u,v,t)}) = 0$, and $\chi'(g_{(u,v,t)}v) = 1$ for all $2 \leq t \leq k$ and all $\{u, v\} \in P \setminus Q$. One can verify that this coloring indeed makes all the pairs in P total rainbow k -connected.

On the other hand, any 3-total-coloring of G' that makes all the pairs in P total rainbow k -connected indeed makes all the pairs in Q total rainbow k -connected in G , because G' contains no path of length 2 between any pair in Q that is not contained in G . Note that there exactly exist k disjoint total rainbow paths between any pair in $P \setminus Q$. For any $e \in \hat{E}_i$, $i \in \{1, 2\}$, from the set of pairs $\left\{ \{b_1, b_2\}, \{b_i, c_e^j\}, \{f_e^j, c\}, \{f_e^j, e^j\}, \{d_e^j, c_e^j\}, \{d_e^j, e^{(3-j)}\} : j \in \{1, 2\} \right\}$, we have $\chi'(b_1c) \neq \chi'(b_2c)$, $\chi'(e) = \chi'(c_e^je^j) = \chi'(c_e^jc) = \chi'(b_{(3-i)}c)$ and $\chi'(e) \notin \{\chi'(e^1), \chi'(e^2)\}$ for $j \in \{1, 2\}$. Hence the coloring χ' of G' not only provides a 3-total-coloring χ of G that makes all the pairs in Q are total rainbow k -connected, but it also make sure that χ extends the original partial coloring $\pi_{\hat{\chi}} = (\hat{E}_1, \hat{E}_2)$ and $\chi(e) \notin \{\chi(e^1), \chi(e^2)\}$ for all $e = e^1e^2 \in \hat{E}$. \square

Lemma 2.5. $3\text{-SAT} \preceq \text{Problem 3}$.

Proof. Given a 3CNF formula $\phi = \bigwedge_{i=1}^m c_i$ over variables $\{x_1, x_2, \dots, x_n\}$, we construct a graph $G = (V, E)$, a partial 2-edge coloring suppose w.l.o.g that $\hat{\chi} : \hat{E} \rightarrow \{0, 1\}$, and

a set of pairs $Q \subseteq (V \times V)$ where Q contains nonadjacent vertex pairs such that there is an extension χ of $\hat{\chi}$ that makes all the pairs in Q total rainbow k -connected and $\chi(e) \notin \{\chi(u), \chi(v)\}$ for all $e = uv \in \hat{E}$ if and only if ϕ is satisfiable. We define G as follows:

$$V(G) = \{c_t : t \in [m]\} \cup \{c_t^j, t \in [m], 2 \leq j \leq k\} \cup \{x_i : i \in [n]\} \cup \{s\}$$

and

$$E(G) = \{c_t x_i : x_i \in c_t \text{ in } \phi\} \cup \{s x_i : i \in [n]\} \cup \left\{ \{s c_t^j, c_t^j c_t\} : t \in [m], 2 \leq j \leq k \right\}.$$

Now we define the set of pairs Q as follows:

$$Q = \left\{ \{s, c_t\} : t \in [m] \right\}.$$

Finally we define the partial 2-edge coloring $\hat{\chi}$ as follows:

$$\hat{\chi}(c_t x_i) = \begin{cases} 0 & \text{if } x_i \text{ is positive in } c_t, \\ 1 & \text{if } x_i \text{ is negative in } c_t. \end{cases}$$

On one hand, if ϕ is satisfiable with a truth assignment over $\{x_1, x_2, \dots, x_n\}$, we extend $\hat{\chi}$ to χ as follows: $\chi(v) = 2$ for all $v \in V$; $\chi(s c_t^j) = 0$, and $\chi(c_t^j c_t) = 1$ for all $t \in [m]$ and all $2 \leq j \leq k$; $\chi(s x_i) = x_i$ for all $i \in [n]$. One can verify χ is as desired. On the other hand, suppose that χ is as desired as above. Note that for any c_t , there must exist a total rainbow path $s x_i c_t$ by some vertex x_i . Set such $x_i = \{\chi(s x_i), \chi(x_i)\} \cap \{0, 1\}$ which can make c_t true. One can verify ϕ is satisfiable. \square

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